

ISOTOPIC CONVERGENCE THEOREM

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ABSTRACT

When approximating a space curve, it is natural to consider whether the knot type of the original curve is preserved in the approximant. This preservation is of strong contemporary interest in computer graphics and visualization. We establish a criterion to preserve knot type under approximation that relies upon pointwise convergence and convergence in total curvature.

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1. Introduction

Curve approximation has a rich history, where the Weierstrass Approximation Theorem is a classical, seminal result [23]. Curve approximation algorithms typically do not include any guarantees about retaining topological characteristics, such as ambient isotopic equivalence. One may easily obtain a sequence of non-trivial knots converging pointwise to a circle, with the knotted portions of the sequence becoming smaller and smaller. These non-trivial knots will never be ambient isotopic to the circle. However, ambient isotopic equivalence is a fundamental concern in knot theory. Moreover, it is a theoretical foundation for curve approximation algorithms in computer graphics and visualization.

So a natural question is what criterion will guarantee ambient isotopic equivalence for curve approximation? The answer is that, besides pointwise convergence, an additional hypothesis of convergence in total curvature will be sufficient, as we shall prove. An example is shown by Figure 1.

Figure 1(a) shows a knotted curve (yellow) which is a trefoil, where this curve is a spline initially defined by an unknotted PL curve (purple), called a control polygon. This PL curve is often treated as the initial approximation of the spline curve. A

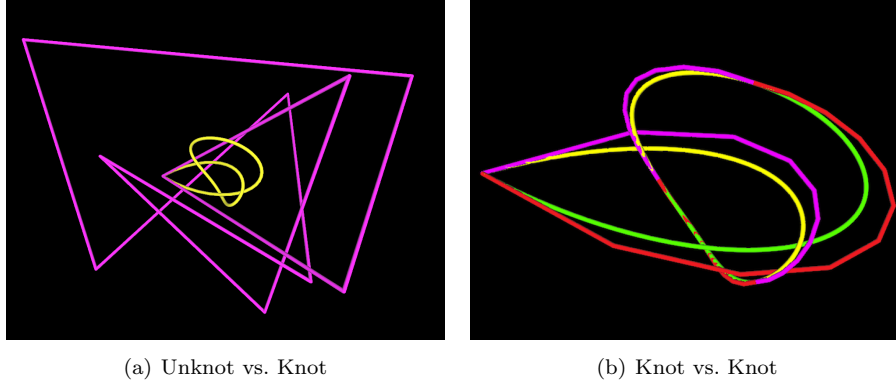


Fig. 1. Ambient isotopic approximation

standard algorithm, called subdivision [6], is used to generate new PL curves that more closely approximate the spline curve. Figure 1(b) shows an ambient isotopic approximation generated by subdivision, as this PL approximation is a trefoil.

There are three main theorems presented. All have a hypothesis of a sequence of curves converging to another smooth curve \mathcal{C} . In Theorem 4.6, the elements of the sequence are PL inscribed curves. In Theorem 5.3 and 7.8, the class of curves is generalized to any piecewise C^2 curves, with the first being a technical result about a lower bound for the total curvature of elements in some tail of the sequence. These first two results are used to provide the main result Theorem 7.8, showing that pointwise convergence and convergence in total curvature over this richer class of piecewise C^2 curves produce a tail of elements that are ambient isotopic to \mathcal{C} .

2. Related Work

The Isotopic Convergence Theorem presented here is motivated by the question about topological integrity of geometric models in computer graphics and visualization. But it is a general and pure theoretical result, dealing with the fundamental equivalence relation in knot theory, which may be applied, but extends beyond the limit of any specific applications.

The preservation of topology in computer graphics and visualization has previously been articulated in two primary applications [9]:

- (1) preservation of isotopic equivalence by approximations; and
- (2) preservation of isotopic equivalence during dynamic changes, such as protein unfolding.

The publications [1,2,15,18] are among the first that provided algorithms to ensure an ambient isotopic approximation. The paper [14] provided existence criteria for a PL approximation of a rational spline curve, but did not include any specific

algorithms.

Recent progress was made for the class of Bézier curves, by providing stopping criteria for subdivision algorithms to ensure ambient isotopic equivalence for Bézier curves of any degree n [10], extending the previous work of [18], that had been restricted to degree less than 4. This extension is based on theorems and sophisticated techniques on knot structures.

This work here extends to a much broader class of curves, piecewise C^2 curves, where there is no restriction on approximation algorithms. Because of its generality, this pure mathematical result is potentially applicable to both theoretical and practical areas.

There exist results in the literature showing ambient isotopy from a different point of view [4,24]. Precisely, there is an upper bound on distance and an upper bound on angles between corresponding points for two curves. If the corresponding distances and angles are within the upper bounds, then they are ambient isotopic.

Milnor [16] defined the total curvature for a C^2 curve using inscribed PL curves. The extension of the definition to piecewise C^2 curves can be trivially done. Consequently, Fenchel's Theorem can be applied to piecewise C^2 curves, as we need here.

Milnor [16] also proved the ambient isotopy between a given C^2 curve and the inscribed curves. That is a similar version of Theorem 4.6 presented here. That result was recently generalized to finite total curvature knots [4]. The application to graphs was also established recently [7]. Our proof here indicates an upper bound on distance and an upper bound on total curvature for ensuring the isotopy, which leads to the formulation of algorithms.

3. Preliminaries

Use \mathcal{C} to denote a compact, regular, C^2 , simple, parametric, space curve. Let $\{C_i\}_1^\infty$ denote a sequence of piecewise C^2 , parametric curves. Suppose all curves are parametrized on $[0, 1]$, that is, $\mathcal{C} = \mathcal{C}(t)$ and $C_i = C_i(t)$ for $t \in [0, 1]$. Denote the sub-curve of \mathcal{C} corresponding to $[a, b] \subset [0, 1]$ as $\mathcal{C}_{[a,b]}$, and similarly use $C_{i[a,b]}$ for C_i . Denote total curvature as a function $T_\kappa(\cdot)$.

3.1. Total curvatures of piecewise C^2 curves

Definition 3.1 (Exterior angles of PL curves). [16] The *exterior angle* between two oriented line segments $\overrightarrow{P_{m-1}P_m}$ and $\overrightarrow{P_mP_{m+1}}$, is the angle between the extension of $\overrightarrow{P_{m-1}P_m}$ and $\overrightarrow{P_mP_{m+1}}$, as shown in Figure 2(a). Let the measure of the exterior angle to be α_m satisfying:

$$0 \leq \alpha_m \leq \pi.$$

This definition naturally generalizes to any two vectors, \vec{v}_1 and \vec{v}_2 , by joining these vectors at their initial points, while denoting the measure between them as $\eta(\vec{v}_1, \vec{v}_2)$,

as indicated in Figure 2(b).

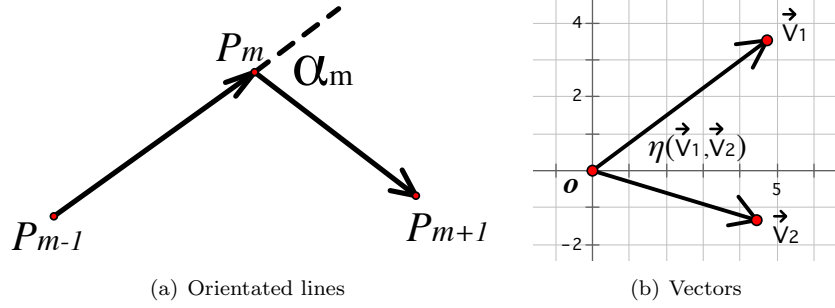


Fig. 2. An exterior angle

The concept of exterior angle is used to unify the concept of total curvature for curves that are PL or differentiable.

Definition 3.2 (Total curvatures of PL curves). [16] The total curvature of a PL curve, is the sum of the exterior angles.

Definition 3.3 (Total curvatures of C^2 curves). [16] The curvature of a C^2 curve $C(t)$ parametrized on $[a, b]$ is given by

$$\kappa(t) = \frac{\|C'(t) \times C''(t)\|}{\|C'(t)\|^3}, \quad t \in [a, b]. \quad (3.1)$$

Its total curvature is the integral: $\int_a^b |\kappa(t)| dt$.

Definition 3.4 (Exterior angles of piecewise C^2 curves). For a piecewise C^2 curve $\gamma(t)$, define the exterior angle at some t_i to be the exterior angle formed by $\gamma'(t_i-)$ and $\gamma'(t_i+)$ where

$$\gamma'(t_i-) = \lim_{h \rightarrow 0} \frac{\gamma(t_i) - \gamma(t_i - h)}{h},$$

and

$$\gamma'(t_i+) = \lim_{h \rightarrow 0} \frac{\gamma(t_i + h) - \gamma(t_i)}{h}.$$

Definition 3.5.^a [Total curvatures of piecewise C^2 curves] Suppose that a piecewise C^2 curve $\phi(t)$ (regular at the C^2 points) is not C^2 at finitely many parameters t_1, \dots, t_n . Denote the sum of the total curvatures of all the C^2 sub-curves as $T_{\kappa 1}$, and the sum of exterior angles at t_1, \dots, t_n as $T_{\kappa 2}$. Then the total curvature of $\phi(t)$ is $T_{\kappa 1} + T_{\kappa 2}$.

^aThis is similarly defined in a recent paper [7].

3.2. Definitions of convergence

Definition 3.6. We say that $\{C_i\}_1^\infty$ converges to \mathcal{C} in *parametric measure distance* if for any $\epsilon > 0$, there exists an integer N such that $\max_{t \in [0,1]} |C_i(t) - \mathcal{C}(t)| < \epsilon$ for all $i \geq N$.

Remark 3.7. For compact curves, this convergence in parametric measure distance is equivalent to pointwise convergence.

Definition 3.8. [20] Let X and Y be two non-empty subsets of a metric space (M, d) . We define their Hausdorff distance $\mu(X, Y)$ by

$$\max\left\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\right\}.$$

Remark 3.9. By the definition of Hausdorff distance, the pointwise convergence implies the convergence in Hausdorff distance.

Definition 3.10. We say that $\{C_i\}_1^\infty$ converges to \mathcal{C} in total curvature if for any $\epsilon > 0$, there exists an integer N such that $|T_\kappa(C_i) - T_\kappa(\mathcal{C})| < \epsilon$ for all $i \geq N$. We designate this property as *convergence in total curvature*.

Definition 3.11. We say that $\{C_i\}_1^\infty$ uniformly converges to \mathcal{C} in total curvature if for any $[t_1, t_2] \subset [0, 1]$ and $\forall \epsilon > 0$, there exists an integer N such that whenever $i \geq N$, $|T_\kappa(C_{i[t_1, t_2]}) - T_\kappa(\mathcal{C}_{[t_1, t_2]})| < \epsilon$. We designate this property as *uniform convergence in total curvature*.

Remark 3.12. Uniform convergence in total curvature implies convergence in total curvature. But the converse is not true.

4. Isotopic Convergence of Inscribed PL Curves

We will use the concept of PL inscribed curves as previously defined [16].

Definition 4.1. A closed PL curve L with vertices v_1, v_2, \dots, v_m is said to be inscribed in curve $\mathcal{C}(t)$ if there is a sequence $\{t_j\}_1^m$ of parameter values such that $v_i = \mathcal{C}(t_j)$ for $j = 1, 2, \dots, m$. We parametrize L over $[0, 1]$, denoted as $L(t)$, by

$$L(t_j) = v_j \text{ for } j = 0, 1, \dots, m$$

and $L(t)$ interpolates linearly between vertices.

The previously established results [16, Theorem 2.2] and [24, Proposition 3.1] showed that a sequence of finer and finer inscribed PL curves will converge in total curvature. The uniform convergence in total curvature follows easily. For the sake of completeness, we present the proof here.

Lemma 4.2. For a piecewise C^2 curve $\gamma(t)$ parametrized on $[0, 1]$ (which is regular at all C^2 points), a sequence $\{L_i\}_1^\infty$ of inscribed PL curves can be chosen such that $\{L_i\}_1^\infty$ pointwise converges to γ and uniformly converges to γ in total curvature.

Proof. We first take the end points $\gamma(t_0) = \gamma(0)$ and $\gamma(t_n) = \gamma(1)$. And then select^b the points where γ fails to be C^2 . Denoted these points as $\{\gamma(t_0), \gamma(t_1), \dots, \gamma(t_{n-1}), \gamma(t_n)\}$. We then compute midpoints: $\gamma(\frac{t_j+t_{j+1}}{2})$ for $j \in \{0, 1, \dots, n-1\}$ to form L_2 which is determined by vertices:

$$\{\gamma(t_0), \gamma(\frac{t_0+t_1}{2}), \gamma(t_1), \dots, \gamma(t_{n-1}), \gamma(\frac{t_{n-1}+t_n}{2}), \gamma(t_n)\}.$$

Continuing this process, we obtain a sequence $\{L_i\}_1^\infty$ of inscribed PL curves.

Suppose the set of vertices of L_i is $\{v_{i,k} = \gamma(t_{i,k})\}$, for some finitely many parameter values $t_{i,k}$. Use uniform parametrization [19] for L_i such that $v_{i,k} = L_i(t_{i,k})$, and points between each pair of consecutive vertices are interpolated linearly. Note first that this process implies that $\{L_i\}_1^\infty$ pointwise converges to \mathcal{C} . For the uniform convergence in total curvature, consider the following:

- (1) Consider each t_j where γ fails to be C^2 . Denote the parameters of two vertices of L_i adjacent to $L_i(t_j)$ as t_{j1}^i and t_{j2}^i . Note that $\lim_{i \rightarrow \infty} t_{j1}^i = \lim_{i \rightarrow \infty} t_{j2}^i = t_j$. This implies that the slope of $\overrightarrow{L_i(t_{j1}^i)L_i(t_j)}$ and the slope of $\overrightarrow{L_i(t_{j2}^i)L_i(t_j)}$ go to $\gamma'(t_j-)$ and $\gamma'(t_j+)$ respectively. This shows that

$$\lim_{i \rightarrow \infty} \eta(\overrightarrow{L_i(t_{j1}^i)L_i(t_j)}, \overrightarrow{L_i(t_{j2}^i)L_i(t_j)}) = \eta(\gamma'(t_j-), \gamma'(t_j+)).$$

- (2) Consider a pair of parameters $t_{i,1}$ and $t_{i,2}$ of any two consecutive vertices of L_i . Denote the corresponding PL curve as $L_{i[t_{i,1}, t_{i,2}]}$. Note that the corresponding sub-curve $\gamma_{i[t_{i,1}, t_{i,2}]}$ of γ is C^2 , since the parameters where γ is not C^2 have been selected as vertices. Provided that $\lim_{i \rightarrow \infty} |t_{i,1} - t_{i,2}| = 0$, the proof of Theorem [16, Theorem 2.2] shows that $T_\kappa(L_{i[t_{i,1}, t_{i,2}]}) \rightarrow T_\kappa(\gamma_{[t_{i,1}, t_{i,2}]})$ as $i \rightarrow \infty$.

By Definition 3.5, the above (1) and (2) together imply that the total curvature of any sub-curve of L_i converges to the corresponding sub-curve of γ (either C^2 or just piecewise C^2), which is the uniform convergence in total curvature (Definition 3.11). \square

Since uniform convergence in total curvature implies convergence in total curvature (Definition 3.11), the corollary below follows immediately.

Corollary 4.3. [16, Theorem 2.2] [24, Proposition 3.1] *For \mathcal{C} , a sequence $\{L_i\}_1^\infty$ of inscribed PL curves can be chosen such that $\{L_i\}_1^\infty$ converges to \mathcal{C} pointwise and in total curvature.*

Theorem 4.4 (Fenchel's Theorem). [16] *The total curvature of a closed curve is at least 2π , with equality holding if and only if the curve is convex.*

^bAcute readers may find later that this choice of points is sufficient for this lemma, but not necessary. This choice is for ease of exposition.

Lemma 4.5. *Denote the plane normal to \mathcal{C} at some $t_0 \in (0, 1)$ as $\Pi(t_0)$. Consider two sub-curves $\mathcal{C}_{[t_0-u]}$ and $\mathcal{C}_{[t_0+v]}$ for some $u \in (0, t_0)$ and $v \in (t_0, 1)$. If both $T_\kappa(\mathcal{C}_{[t_0-u]}) < \frac{\pi}{2}$ and $T_\kappa(\mathcal{C}_{[t_0+v]}) < \frac{\pi}{2}$, then these two sub-curves $\mathcal{C}_{[t_0-u]}$ and $\mathcal{C}_{[t_0+v]}$ are separated by $\Pi(t_0)$ except at $\mathcal{C}(t_0)$.*

Proof. Denote the point $\mathcal{C}(t_0)$ as a . Suppose that the conclusion is false, then either $\mathcal{C}_{[t_0-u]}$ or $\mathcal{C}_{[t_0+v]}$ intersects $\Pi(t_0)$ other than at a . Assume without loss of generality that $\mathcal{C}_{[t_0+v]} \cap \Pi(t_0)$ contains another point, denoted as b . Then the sub-curve $\mathcal{C}_{[t_0+v]}$ and the line segment \overline{ab} form a closed curve $\mathcal{C}_{[t_0+v]} \cup \overline{ab}$. So $T_\kappa(\mathcal{C}_{[t_0+v]} \cup \overline{ab}) \geq 2\pi$ by Theorem 4.4.

Denote the exterior angles at a and b as α and β respectively. Then $\alpha = \frac{\pi}{2}$ since $\Pi(t_0)$ is normal to $\mathcal{C}'(t_0)$. By Definition 3.1, $\beta \leq \pi$. By Definition 3.5 we have

$$T_\kappa(\mathcal{C}_{[t_0+v]} \cup \overline{ab}) = T_\kappa(\mathcal{C}_{[t_0+v]}) + \alpha + \beta \leq T_\kappa(\mathcal{C}_{[t_0+v]}) + \frac{\pi}{2} + \pi.$$

So

$$T_\kappa(\mathcal{C}_{[t_0+v]}) + \frac{\pi}{2} + \pi \geq 2\pi.$$

Therefore

$$T_\kappa(\mathcal{C}_{[t_0+v]}) \geq \frac{\pi}{2},$$

which is a contradiction. \square

Theorem 4.6 below is restricted to “inscribed PL curves”. The general theorem of “piecewise C^2 curves, either inscribed or not” will be established later in Theorem 7.8.

Theorem 4.6. *For any sequence $\{L_i\}_1^\infty$ of inscribed PL curves that pointwise converges to \mathcal{C} and uniformly converges to \mathcal{C} in total curvature, a positive integer N can be found as below such that for all $i > N$, L_i is ambient isotopic to \mathcal{C} .*

Proof. For \mathcal{C} , there is a non-singular tubular surface^c of radius r [14].

Pointwise convergence and the uniform convergence in total curvature imply that there exists a positive integer N such that for an arbitrary $i > N$:

- (1) The PL curve L_i lies inside of the tubular surface of radius r ; and
- (2) Denote the set of vertices of L_i as $\{v_j\}_{j=0}^n$. Suppose the sub-curve of \mathcal{C} between two arbitrary consecutive vertices v_j and v_{j+1} as \mathcal{A}_j , for $j = 0, \dots, n-1$. Then since the total curvature of $\overrightarrow{v_j v_{j+1}}$ is 0, the total curvature of \mathcal{A}_j can be less than $\frac{\pi}{2}$.

^cWe use the terminology of *tubular surface* as generalization from the recent usage [14] regarding the classically defined *pipe surface* [17].

Lemma 4.5 implies that all such sub-curves \mathcal{A}_j are separated by normal planes except the connection points. The facts about fitting inside a tubular surface and separation by normal planes provide a sufficient condition [14] for L_i being ambient isotopic to \mathcal{C} . \square

Remark 4.7. The paper [14] provides the computation of the radius r only for rational spline curves. However, the method of computing r is similar for other compact, regular, C^2 , and simple curves, that is, setting

$$r < \min\left\{\frac{1}{\kappa_{\max}}, \frac{d_{\min}}{2}, r_{\text{end}}\right\},$$

where κ_{\max} is the maximum of the curvatures, d_{\min} is the minimum separation distance, and r_{end} is the maximal radius around the end points that does not yield self-intersections.

5. Pointwise Convergence

Pointwise convergence provides a lower bound of the total curvatures of approximants (Theorem 5.3). The proof relies upon showing this for PL curves first (Lemma 5.2). The technique used here is the well known “2D push” [3]. It is sufficient here to consider a specialized type of push, designated, below, as a *median push*.

Definition 5.1. Assume that triangle $\triangle ABC$ has non-collinear vertices A, B and C . Push a vertex, say B , along the corresponding median of the triangle to the midpoint of the side AC . We call this specific kind of “2D push”, a median push.

Lemma 5.2. Let $\{L_i\}_{i=1}^{\infty}$ be a sequence of PL curves parametrized on $[0, 1]$ and L be a PL curve parametrized on $[0, 1]$. If $\{L_i\}_{i=1}^{\infty}$ pointwise converges to L , then for $\forall \epsilon > 0$, there exists an integer N such that $T_{\kappa}(L_i) > T_{\kappa}(L) - \epsilon$ for all $i \geq N$.

Proof. For an arbitrary vertex v of L , suppose $v = L(t_v)$ for some $t_v \in [0, 1]$. Let B_v be a closed ball centered at v . Since L is a compact PL curve, we can choose the radius of B_v small enough such that:

- (1) the ball B_v contains only the single vertex v of L ; and
- (2) it intersects only the two line segments of L which are connected at v . Denote these intersections as $u = L(t_u)$ and $w = L(t_w)$ for some $t_u, t_w \in [0, 1]$. Then u, w and v together form a triangle $\triangle uvw$.

Let $u_i = L_i(t_u)$, $v_i = L_i(t_v)$ and $w_i = L_i(t_w)$. Denote the exterior angle of the triangle $\triangle uvw$ at v as $\eta(v)$, and correspondingly the exterior angle of $\triangle u_i v_i w_i$ at v_i as $\eta(v_i)$. (Note that $\eta(v)$ is not necessarily equal to the exterior angle of L at v . Similarly for $\eta(v_i)$.) By the pointwise convergence we have that the triangle

$\triangle u_i v_i w_i$ converges to $\triangle uvw$. So $\eta(v_i)$ converges to $\eta(v)$. That is, for $\forall \epsilon' > 0$ there exists an N such that $\eta(v_i) > \eta(v) - \epsilon'$ for all $i \geq N$.

Consider the PL sub-curve of L_i lying in B_v and denote its total curvature as $T_\kappa(L_i \cap B_v)$. This PL sub-curve of L_i can be reduced by median pushes to $\triangle u_i v_i w_i$. [16, Lemma 1.1, Corollary 1.2] implies that $T_\kappa(L_i \cap B_v) \geq \eta(v_i)$. So for $i \geq N$,

$$T_\kappa(L_i \cap B_v) > \eta(v) - \epsilon'. \quad (5.1)$$

Denote the set of vertices of L as V . Then $T_\kappa(L) = \sum_{v \in V} \eta(v)$. Note that $T_\kappa(L_i) \geq \sum_{v \in V} T_\kappa(L_i \cap B_v)$. So Inequality 5.1 implies that

$$T_\kappa(L_i) \geq \sum_{v \in V} T_\kappa(L_i \cap B_v) > \sum_{v \in V} \eta(v) - \epsilon' n = T_\kappa(L) - \epsilon' n$$

where n is the number of vertices of L . Let $\epsilon' = \frac{\epsilon}{n}$, then we complete the prove. \square

Theorem 5.3. *If $\{C_i\}_1^\infty$ pointwise converges to \mathcal{C} , then for $\forall \epsilon > 0$, there exists an integer N such that $T_\kappa(C_i) > T_\kappa(\mathcal{C}) - \epsilon$ for all $i \geq N$.*

Proof. By Lemma 4.2, we can use inscribed PL curves to approximate $\{C_i\}_1^\infty$ and \mathcal{C} , such that the approximations converge pointwise and in total curvature. Then apply the Lemma 5.2 to these inscribed PL curves. Since these inscribed PL curves converge pointwise and in total curvature to $\{C_i\}_1^\infty$ and \mathcal{C} respectively, the desired conclusion follows. \square

6. Uniform convergence in total curvature

Convergence in total curvature is weaker than uniform convergence in total curvature. But pointwise convergence and convergence in total curvature together imply uniform convergence in total curvature, which is shown by Lemma 6.1 below.

Lemma 6.1. *If $\{C_i\}_1^\infty$ converges to a C^2 curve \mathcal{C} pointwise and in total curvature, then $\{C_i\}_1^\infty$ uniformly converges to \mathcal{C} in total curvature.*

Proof. Assume not, then there exist a subset $[t_1, t_2] \subset [0, 1]$ and a $\tau > 0$ such that for any integer N , there is a $i \geq N$ such that $|T_\kappa(C_{i[t_1, t_2]}) - T_\kappa(\mathcal{C}_{[t_1, t_2]})| > \tau$, that is $T_\kappa(C_{i[t_1, t_2]}) > T_\kappa(\mathcal{C}_{[t_1, t_2]}) + \tau$ or $T_\kappa(C_{i[t_1, t_2]}) < T_\kappa(\mathcal{C}_{[t_1, t_2]}) - \tau$. The latter is precluded by Theorem 5.3. Therefore

$$T_\kappa(C_{i[t_1, t_2]}) > T_\kappa(\mathcal{C}_{[t_1, t_2]}) + \tau. \quad (6.1)$$

Consider the sequence of the sub-curves of $\{C_i\}_1^\infty$ restricted to the complement $[t_1, t_2]^c$ of $[t_1, t_2]$, and denote it as $\{C_{i[t_1, t_2]^c}\}_1^\infty$. By theorem 5.3, for $\frac{\tau}{2}$, there exists an integer, say M such that for all $i \geq M$,

$$T_\kappa(C_{i[t_1, t_2]^c}) > T_\kappa(\mathcal{C}_{[t_1, t_2]^c}) - \frac{\tau}{2}. \quad (6.2)$$

Note that $T_\kappa(C_i) \geq T_\kappa(C_{i[t_1, t_2]}) + T_\kappa(C_{i[t_1, t_2]^c})$. So Equations 6.1 and 6.2 imply that there is a $i \geq M$ so that

$$T_\kappa(C_i) \geq T_\kappa(C_{i[t_1, t_2]}) + T_\kappa(C_{i[t_1, t_2]^c}) > T_\kappa(\mathcal{C}_{[t_1, t_2]}) + T_\kappa(\mathcal{C}_{[t_1, t_2]^c}) + \frac{\tau}{2}.$$

Since \mathcal{C} is C^2 , $T_\kappa(\mathcal{C}_{[t_1, t_2]}) + T_\kappa(\mathcal{C}_{[t_1, t_2]^c}) = T_\kappa(\mathcal{C})$. Therefore we get

$$T_\kappa(C_i) \geq T_\kappa(\mathcal{C}) + \frac{\tau}{2},$$

which contradicts the convergence in total curvature. \square

7. Isotopic Convergence

For a C^2 compact curve \mathcal{C} , we shall, without loss of generality (Theorem 4.6), consider a sequence $\{L_i\}_1^\infty$ of PL curves (instead of piecewise C^2 curves) as its approximation. We shall divide \mathcal{C} into finitely many sub-curves, and reduce the corresponding sub-curves of L_i to line segments, by median pushes, so as to preserve isotopic equivalence. The line segments generated by the pushes form a polyline. We shall then prove the polyline is ambient isotopic to \mathcal{C} .

To get to the major theorem, we need to first establish some preliminary topological results. We use $CH(\cdot)$ to denote the convex hull of a set.

Lemma 7.1. *Let X and Y be compact subspaces of an Euclidean space \mathbb{R}^d . If $X \cap Y = \emptyset$, then Y can be subdivided into finitely many subsets, denoted as $Y_1, \dots, Y_i, \dots, Y_m$ for some $m > 0$, such that $CH(Y_i) \cap X = \emptyset$ for each i .*

Proof. Since X is compact, for $\forall y \in Y$, $\inf_{x \in X} \|x - y\| > 0$, and hence \exists an open ball $B_y \subset \mathbb{R}^d$ of y such that $B_y \cap X = \emptyset$. Since Y is compact, among these open balls, there are finitely many, denoted by B_{y_1}, \dots, B_{y_m} such that $Y \subset \bigcup_{i=1}^m B_{y_i}$.

Let $Y_i = Y \cap B_{y_i}$ for each $i = 1, \dots, m$ so that

$$CH(Y_i) = CH(Y \cap B_{y_i}) \subset CH(B_{y_i}) = B_{y_i}.$$

Thus, for each i , we have $CH(Y_i) \cap X = \emptyset$. \square

As we mentioned before, for a simple C^2 curve \mathcal{C} , there is a non-singular tubular surface of radius r (Remark 4.7). This surface determines a tubular neighborhood of \mathcal{C} , denoted as $\Gamma_{\mathcal{C}}$. Denote a sub-curve of \mathcal{C} as \mathcal{C}^k , and the corresponding tubular neighborhood of \mathcal{C}^k as Γ^k .

Lemma 7.2. *The compact curve \mathcal{C} can be divided into finitely many sub-curves, denoted as $\mathcal{C}^1, \dots, \mathcal{C}^k, \dots, \mathcal{C}^n$ for some $n > 0$, such that*

- $T_\kappa(\mathcal{C}^k) < \frac{\pi}{2}$; and
- $CH(\mathcal{C}^k) \subset \Gamma^k$.

Proof. By Lemma 7.1, \mathcal{C} can be partitioned into finitely many non-empty sub-curves, each which is disjoint from $S_r(\mathcal{C})$. Since \mathcal{C} is also of finite total curvature, we can denote these sub-curves as $\mathcal{C}^1, \dots, \mathcal{C}^k, \dots, \mathcal{C}^n$ for some $n > 0$, such that for each $k = 1, \dots, n$, $T_\kappa(\mathcal{C}^k) < \frac{\pi}{2}$ and $CH(\mathcal{C}^k) \cap S_r(\mathcal{C}) = \emptyset$.

Consider \mathcal{C}^k for an arbitrary $k = 1, \dots, n$ and denote the distinct normal planes at the endpoints of \mathcal{C}^k by Π_1, Π_2 , respectively. Denote the closed convex subspace of \mathbb{R}^3 that contains \mathcal{C}^k and is bounded by Π_1 and Π_2 as H^k . It is clear that $CH(\mathcal{C}^k) \subset H^k$, but since $CH(\mathcal{C}^k) \cap S_r(\mathcal{C}) = \emptyset$, we have that $CH(\mathcal{C}^k) \subset \Gamma^k$. \square

For $k = 1, \dots, n$, let $[t_{k-1}, t_k]$ be the subinterval whose image is \mathcal{C}^k , with corresponding Γ^k . Let ϵ be real valued such that

$$0 < \epsilon < \min_{k \in \{0, \dots, n\}} \frac{|t_k - t_{k-1}|}{2}.$$

We extend^d $[t_{k-1}, t_k]$ to $[t_{k-1} - \epsilon, t_k + \epsilon]$, and denote the tubular neighborhood corresponding to the extended subinterval as Γ_ϵ^k , then Γ_ϵ^k only intersects Γ_ϵ^{k+1} and Γ_ϵ^{k-1} for each k .

For a sequence of PL curves $\{L_i\}_1^\infty$ converging to \mathcal{C} pointwise and in total curvature, denote the sub-curve of L_i corresponding (with the same parameters) to \mathcal{C}^k as L_i^k . Denote the end points of L_i^k by u_i^{k-1} and u_i^k , with the corresponding end points of \mathcal{C}^k by v^{k-1} and v^k .

Lemma 7.3. *A large positive integer N can be found such that whenever $i \geq N$, for each k , we have*

- (1) $T_\kappa(L_i^k) < \frac{\pi}{2}$;
- (2) $CH(L_i^k) \subset \Gamma_\epsilon^k$; and
- (3) $|u_i^k - v^k| < \frac{r}{2}$ and $\mu(\overline{u_i^{k-1}u_i^k}, \mathcal{C}^k) < \frac{r}{2}$, where $\mu(\cdot)$ refers to the Hausdorff distance.

Proof. The first condition follows from the uniform convergence in total curvature (Lemma 6.1), and the second and third follow from pointwise convergence. \square

Now we are ready to reduce each L_i^k to the segment $\overline{u_i^{k-1}u_i^k}$ by median pushes. In order to prove there is no self-intersection of L_i during the pushes, we present two lemmas below. The following lemma was established by a recent preprint [11]. For the sake of completeness, we give the sketch of the proof here.

Lemma 7.4 (Non-self-intersection criteria). [11] *Let $P = (P_0, P_1, \dots, P_n)$ be an open PL curve in \mathbb{R}^3 . If $T_\kappa(P) = \sum_{j=1}^{n-1} \alpha_j < \pi$, then P is simple.*

Proof. Assume to the contrary that P is self-intersecting. Then there must exist at least one closed loop. Consider one closed loop. By Fenchel's theorem, the total

^dIf \mathcal{C} is open and $t_{k-1} = 0$ or $t_k = 1$, consider $[0, t_k + \epsilon]$ or $[t_{k-1} - \epsilon, 1]$.

curvature of the closed loop is at least 2π . The total curvature is the sum of the exterior angles, among which at most one angle is not counted as an exterior angle of P . But an exterior angle is less than π . So the total curvature of P is at least $2\pi - \pi = \pi$, which is a contradiction. \square

Milnor [16] showed the total curvature remains the same or decreases “after” deforming a triangle to a line segment, and this can be trivially extended to show that the total curvature remains the same or decreases “during” the whole process of deforming a triangle to a line segment, as expressed in Lemma 7.5.

Lemma 7.5. *If a vertex of a PL curve in \mathbb{R}^3 undergoes a median push, then the total curvatures of new open PL curves formed during the push remain the same or decrease^e.*

Lemma 7.6. *For each $k = 1, \dots, n$, use median pushes to reduce L_i^k to the line segment $\overline{u_i^{k-1}u_i^k}$. Then during these pushes, L_i remains simple, and hence the resultant PL curve $\bigcup_{k=1}^n \overline{u_i^{k-1}u_i^k}$ is ambient isotopic to the original PL curve L_i .*

Proof. Note that the condition (1) in Lemma 7.3 implies that $T_\kappa(L_i^{k-1} \cup L_i^k) < \pi$ and $T_\kappa(L_i^k \cup L_i^{k+1}) < \pi$. Lemma 7.4 and 7.5 show that the pushed L_i^k does not intersect its neighbors L_i^{k+1} or L_i^{k-1} . Since $CH(L_i^k) \subset \Gamma_\epsilon^k$ (Condition (2)), and Γ_ϵ^k does not intersect Γ_ϵ^j for $j \neq k-1$ or $k+1$, the perturbed L_i^k stays inside Γ_ϵ^k and does not intersect L_i^j for $j \neq k-1$ or $k+1$. Then the conclusion follows. \square

For each $k = 1, \dots, n$, connecting the end points v^{k-1} and v^k of \mathcal{C}^k , we obtain the polyline $\bigcup_{k=1}^n \overline{v^{k-1}v^k}$.

Lemma 7.7. *The polyline $\bigcup_{k=1}^n \overline{v^{k-1}v^k}$ is ambient isotopic to $\bigcup_{k=1}^n \overline{u_i^{k-1}u_i^k}$.*

Proof. Perturb u_i^k to v^k , and the line segments move linearly from $\overline{u_i^{k-1}u_i^k}$ to $\overline{u_i^{k-1}v^k}$, and from $\overline{u_i^k u_i^{k+1}}$ to $\overline{v^k u_i^{k+1}}$. Since $|u_i^k - v^k| < \frac{r}{2}$ and $\mu(\overline{u_i^{k-1}u_i^k}, \mathcal{C}^k) < \frac{r}{2}$ (Condition (3) in Lemma 7.3), the perturbation stays inside Γ_ϵ^k which has a radius r . So during the perturbation, $\overline{u_i^{k-1}, u_i^k}$ and $\overline{u_i^k, u_i^{k+1}}$ do not intersect any line segments of L_i , possibly except their consecutive segments. But note that for each k , $u_i^k, v^k \in \Gamma_\epsilon^k \cap \Gamma_\epsilon^{k+1}$. An easy geometric analysis shows that this restricted area of the perturbation precludes the possibility for $\overline{u_i^{k-1}, u_i^k}$ and $\overline{u_i^k, u_i^{k+1}}$ intersecting their consecutive segments. So the perturbation does not cause intersections, and hence preserves the ambient isotopy. \square

^eThis holds not only for the median push, but also for any push with a trace lying on the interior of a triangle indicated in Definition 5.1.

Below is the general Isotopic Convergence Theorem, while Theorem 4.6 is a special version restricted to “inscribed PL curves”.

Theorem 7.8 (Isotopic Convergence Theorem). *If $\{C_i\}_1^\infty$ converges to \mathcal{C} pointwise and in total curvature, then there exists an integer N such that C_i is ambient isotopic to \mathcal{C} for all $i \geq N$.*

Proof. For each C_i and $\epsilon > 0$, there exists an inscribed PL curve L_i of C_i such that L_i is sufficiently close (bounded by ϵ) to C_i pointwise and in total curvature by Lemma 4.2, and ambient isotopic to C_i by Theorem 4.6. Since ambient isotopy is an equivalence relation [12], we now rely on Theorem 4.6 to consider, without loss of generality, a sequence of PL curves $\{L_i\}_1^\infty$ instead of $\{C_i\}_1^\infty$.

Note that $T_\kappa(\mathcal{C}^k) < \frac{\pi}{2}$ and the polyline $\bigcup_{k=1}^n \overline{v^{k-1}v^k}$ lies inside of the tubular neighborhood (since $CH(\mathcal{C}^k) \subset \Gamma_{\mathcal{C}}$). By the proof of Theorem 4.6 we know that these are sufficient conditions for \mathcal{C} being ambient isotopic to $\bigcup_{k=1}^n \overline{v^{k-1}v^k}$. By the equivalence relation of ambient isotopy, Lemma 7.7 implies that \mathcal{C} is ambient isotopic to $\bigcup_{k=1}^n \overline{u_i^{k-1}u_i^k}$, and Lemma 7.6 further implies that \mathcal{C} is ambient isotopic to L_i . \square

8. Some Conceptual Algorithms and Potential Applications

The Isotopic Convergence Theorem has both theoretical and practical applications. Theoretically, it formulates criteria to show the same knot type in knot theory. Practically, it provides rigorous theoretical foundations to extend current algorithms in computer graphics and visualization to much richer classes of curves than the splines already investigated [8].

The following are the general procedures derived from our Theorem 4.6 and Theorem 7.8. For a specific problem, further algorithmic development will depend upon characteristics of the class of curves. If the curve is “nice” in the sense that the total curvature and the radius of a tubular surface is easy to compute, then it is easy to develop an algorithm. Such “nice” curves include a rational cubic spline parameterized by arc length, for which the total curvatures can be easily computed, and the radius of a tubular surface can be found according to an existing result [14]. Otherwise, for some other curves, the computation of the total curvatures and the radius of a tubular neighborhood may be difficult and is beyond the scope of the details considered here, even while the theorems provide a broad framework within which these subtleties can be considered.

8.1. Using PL knots to represent smooth knots

Based on Lemma 4.2 and Theorem 4.6, a procedure can be designed such that it takes a smooth knot as input and picks finitely many points on it to form an ambient isotopic PL knot. We call this a *PL representation* of the smooth knot.

Recall that two criteria are sufficient for the isotopy between a compact C^2 curve \mathcal{C} and its inscribed PL curve \mathcal{L} :

- (1) Each sub-curve of \mathcal{C} determined by two consecutive vertices of \mathcal{L} has a total curvature less than $\frac{\pi}{2}$.
- (2) The PL curve \mathcal{L} lies inside of the tubular surface for \mathcal{C} with radius r . (This can be achieved by making the Hausdorff distance between \mathcal{L} and \mathcal{C} less than r .)

The Outline of Forming PL Representations:

- (1) Select $\mathcal{C}(0)$ as the initial vertex of \mathcal{L} , denoted as v_0 .
- (2) Set^f $\epsilon = 0.1$. Select^g $t_1 \in [0, 1]$ such that $T_\kappa(\mathcal{C}_{[0, t_1]}) = \frac{\pi}{2} - \epsilon$. Let the second vertex of \mathcal{L} be $\mathcal{C}(t_1)$, denoted as v_1 .
- (3) Similarly pick t_2 to obtain v_2 . Continue until we reach the end point $\mathcal{C}(1)$, denoted as v_n . This process terminates because \mathcal{C} is compact. In the end, we obtain an \mathcal{L} , and sub-curves of \mathcal{C} with total curvatures being less than $\frac{\pi}{2}$.
- (4) Verify if the Hausdorff distance between \mathcal{L} and \mathcal{C} is less than r ; If not, then select midpoints: $\mathcal{C}(\frac{t_j + t_{j+1}}{2})$ for $j \in \{0, 1, \dots, n-1\}$, denoted as $v_{\frac{2j+1}{2}}$, to form a new inscribed PL curve determined by vertices:

$$\{v_0, v_{\frac{1}{2}}, v_1, v_{\frac{3}{2}}, v_2, \dots, v_{n-1}, v_{\frac{2n-1}{2}}, v_n\}.$$

- (5) Repeat 4 until the Hausdorff distance between \mathcal{L} and \mathcal{C} is less than r . This process of selecting midpoints implies the pointwise convergence. So this process terminates.

8.2. Testing isotopic convergence

For a C^2 curve \mathcal{C} and a sequence $\{C_i\}_1^\infty$ of piecewise C^2 curves, where $\{C_i\}_1^\infty$ converges to \mathcal{C} pointwise and in total curvature, we shall design a procedure to determine a positive integer N such that whenever $i \geq N$, C_i is ambient isotopic to \mathcal{C} . Use $T_\kappa(\cdot)$ to denote the total curvature, $CH(\cdot)$ the convex hull, and $\mu(\cdot)$ the Hausdorff distance.

The Outline of Testing Isotopic Convergence:

- (1) Divide \mathcal{C} into sub-curves \mathcal{C}^k for $k = 1, \dots, n$ such that
 - $T_\kappa(\mathcal{C}^k) < \frac{\pi}{2}$; and
 - $CH(\mathcal{C}^k) \subset \Gamma^k$.
- (2) Set $i := 1$.

^fThis ϵ value is not unique. Many others also work.

^gIt is not necessary for $T_\kappa(\mathcal{C}_{[0, t_1]})$ to be exactly $\frac{\pi}{2} - \epsilon$. For efficiency, it is fine to end up with a value not equal to $\frac{\pi}{2} - \epsilon$ as long as it is less than $\frac{\pi}{2}$. This aspect will require a subroutine to be developed that will likely vary over the class of curves considered and this detail is beyond the scope of the current investigation.

- (3) Use the above technique to form a *PL Representation* L_i for the piecewise C^2 curve C_i such that L_i is ambient isotopic to C_i .
- (4) Let L_i^k be the sub-curve of L_i corresponding to C^k . Denote the end points of L_i^k as u_i^{k-1} and u_i^k , and the corresponding end points of C^k as v^{k-1} and v^k . Verify three criteria:
 - $T_\kappa(L_i^k) < \frac{\pi}{2}$;
 - $CH(L_i^k) \subset \Gamma_\epsilon^k$; and
 - $|u_i^k - v^k| < \frac{r}{2}$ and $\mu(\overline{u_i^{k-1}u_i^k}, C^k) < \frac{r}{2}$.
- (5) If these criteria are satisfied, then let $N := i$ and stop. Otherwise let $i := i + 1$ and go to (3).

We know that $\{C_i\}_1^\infty$ pointwise converges to \mathcal{C} and uniformly converges to \mathcal{C} in total curvature (Lemma 6.1), so there exists a finite i such that these three criteria are achieved, which means the above process terminates. By Isotopic Convergence Theorem, we obtain the ambient isotopy.

8.3. A potential application for molecular simulations

It is often of interest to consider geometric models that are perturbed over time. For chemical simulations of macro-molecules, the algorithms in high performance computing (HPC) environments will produce voluminous numerical data describing how the molecule twists and writhes under local chemical and kinetic changes. These are reflected in changed co-ordinates of the geometric model, called perturbations. To produce a scientifically valid visualization, it is crucial that topological artifacts are not introduced by the visual approximations [8]. A primary distinction between the approximations created here in Section 8.1 and those based on Taylor's Theorem [8] is the expression here of the upper bound for total curvature to be less than $\frac{\pi}{2} - \epsilon$.

An earlier perturbation result is limited to PL curves [2]. Other ambient isotopic approximation methods rely upon the curve being a spline [10,18]. The example in the next section needs not be a spline. Our approach provides a more general result of sufficient conditions for both approximation and perturbations which do not change topological features. Precisely, our Isotopic Convergence Theorem implies that as long as the convergence criterion of pointwise convergence and convergence in total curvature is satisfied, then ambient isotopy is preserved.

8.4. A representative example of offset curves

Offset curves are defined as locus of the points which are at constant distant along the normal from the generator curves [13]. It is well-known [22, p. 553] that offsets of spline curves need not be splines. They are widely used in various applications, and the related approximation problems were frequently studied. A literature survey on offset curves and surfaces prior to 1992 was conducted by Pham [21], and another such survey between 1992 and 1999 was given by Maekawa [13]. Here we show a

representative example as a catalyst to ambient isotopic approximations of offset curves.

Let $\mathcal{C}(t)$ be a compact, regular, C^2 , simple, space curve parametrized in $[a, b]$, whose curvature κ never equals 1. Then define an offset curve by

$$\Omega(t) = \mathcal{C}(t) + N(t),$$

where $N(t)$ is the normal vector at t , for $t \in [a, b]$.

For example, let $\mathcal{C}(t) = (2 \cos t, 2 \sin t, t)$ for $t \in [0, 2\pi]$ be a helix, then it is an easy exercise for the reader to verify that the above assumptions of \mathcal{C} are satisfied, with $\kappa = \frac{2}{5}$. Furthermore, it is straightforward to obtain the offset curve $\Omega(t) = (\cos t, \sin t, t)$, which is not a spline.

We first show that $\Omega(t)$ is regular. Let $s(t) = \int_a^t |\mathcal{C}'(t)| dt$ be the arc-length of \mathcal{C} . Then by Frenet-Serret formulas [5] we have

$$\begin{aligned} \Omega'(t) &= \mathcal{C}'(t) + N'(t) \\ &= \frac{ds}{dt}T + (-\kappa T + \tau B)\frac{ds}{dt} = (1 - \kappa)\frac{ds}{dt}T + \tau\frac{ds}{dt}B, \end{aligned}$$

where T and B are the unit tangent vector and binormal vector respectively. Since $T \perp B$, if $(1 - \kappa)\frac{ds}{dt} \neq 0$ then $\Omega'(t) \neq 0$. But $(1 - \kappa)\frac{ds}{dt} \neq 0$ because $\kappa \neq 1$ and $\mathcal{C}(t)$ is regular by the assumption. Thus $\Omega(t)$ is regular.

Now we define a sequence $\{\Omega_i(t)\}_{i=1}^\infty$ to approximate $\Omega(t)$ by setting

$$\Omega_i(t) = \mathcal{C}(t) + \frac{i-1}{i}N(t).$$

It is obvious that $\{\Omega_i(t)\}_{i=1}^\infty$ pointwise converges to $\Omega(t)$. For the convergence in total curvature, note that $\lim_{i \rightarrow \infty} \Omega'_i(t) = \Omega'(t)$, $\lim_{i \rightarrow \infty} \Omega''_i(t) = \Omega''(t)$, and $|\Omega'(t)| \neq 0$ due to the regularity of $\Omega(t)$. Therefore

$$\lim_{i \rightarrow \infty} \frac{\Omega'_i(t) \times \Omega''_i(t)}{|\Omega'_i(t)|^3} = \frac{\Omega'(t) \times \Omega''(t)}{|\Omega'(t)|^3}.$$

The convergence in total curvature follows.

Consequently, by^h the Isotopic Convergence Theorem (Theorem 7.8), we conclude that there exists a positive integer N such that $\Omega_i(t)$ is ambient isotopic to $\Omega(t)$ whenever $i > N$.

9. Conclusion

We derived the Isotopic Convergence Theorem by topological and geometric techniques, as motivated by applications for knot theory, computer graphics, visualization and simulations.

^hSince the example satisfies C^0 and C^1 convergence and the paper [24] shows ambient isotopy under C^0 and C^1 convergence, the ambient isotopy for this example also follows from the previous result [24]. Here our purpose is to use it as a representation to show how the Isotopic Convergence Theorem can be applied.

A future research direction could use the Isotopic Convergence Theorem in knot classification, since it provides a method to pick finitely many points from a given knot, where the set of finitely many points determines the same knot type.

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